Lattice Realizations of the Open Descendants of Twisted Boundary Conditions for $sl(2)$ A-D-E Models

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Abstract

The twisted boundary conditions and associated partition functions of the conformal $sl(2)$ A-D-E models are studied on the Klein bottle and the Möbius strip. The A-D-E minimal lattice models give realization to the complete classification of the open descendants of the $sl(2)$ minimal theories. We construct the transfer matrices of these lattice models that are consistent with non-orientable geometries. In particular, we show that in order to realize all the Klein bottle amplitudes of different crosscap states, not only the topological flip on the lattice but also the involution in the spin configuration space must be taken into account. This involution is the $\mathbb{Z}_2$ symmetry of the Dynkin diagrams which corresponds to the simple current of the Ocneanu algebra.

1 Introduction

The study of conformal field theories on non-orientable surfaces has attracted much attention since the discovery of D-branes [1]. Open descendants [2] is a systematic construction of conformal field theories on non-orientable surfaces from the oriented ones by requiring the theories to satisfy certain consistency conditions [3]. The open descendants of many different models have been studied [4, 5, 6, 7, 8]. Moreover, it was discovered that new crosscap states can be found by the approach of simple current invariance [9, 10, 11]. Conformal defects on non-orientable surfaces were discussed in [12].

On the other hand, it is well known that statistical mechanics models realize conformal field theories in the continuum scaling limit. In particular, the lattice realizations [13, 14] of conformal A-D-E unitary minimal models on a torus with defect lines [15, 16] has been studied. The Ising model on the Klein bottle and Möbius strip has also been investigated [17, 18]. In this article, we study the critical $sl(2)$ A-D-E minimal lattice models on the Klein bottle and on the Möbius strip with twisted boundary conditions. All the lattice realizations

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of the known open descendants for the $sl(2)$ models are obtained in these non-orientable geometries.

The paper is organized as follows. In section 2, we briefly review the results of the boundaries conditions of the $sl(2)$ models on a torus (section 2.1) and on a cylinder (section 2.2). In section 3, we review the open descendants on the Klein bottle and the Möbius strip. In section 3.3, we describe in detail how to compute the crosscap coefficients and to obtain the partition functions on the Klein bottle and the Möbius strip. In section 4, we define the A-D-E lattice models and construct the transfer matrices on the Klein bottle and the Möbius strip. In section 5, we summarize our numerical results.

2 Conformal Boundary Conditions of $sl(2)$ Models

The chiral algebra of an $SU(2)_k$ Wess-Zumino-Witten model is given by the affine algebra $\hat{sl}(2)$ at level k and there exists a finite set $\mathcal{I} = \{1, 2, \ldots, k+1\}$ of irreducible representations $\mathcal{V}_i$, $i \in \mathcal{I}$. The central charge is $c = 3k/(k+2)$ and the conformal weights are $h_j = (j^2-1)/4(k+2)$, $j \in \mathcal{I}$.

It is well known that the modular invariant partition functions of the $SU(2)_k$ WZW models are classified [19] according to the Dynkin diagrams of the classical A-D-E simply laced Lie algebras, where the Coxeter number g is identified with $k+2$.

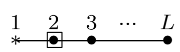
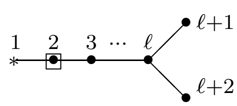
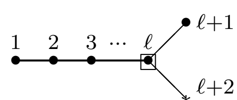
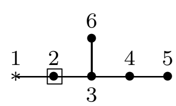
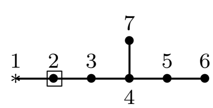
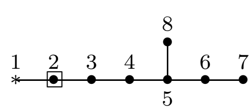
	Graph G	g	Exp(G)	Type/H	Γ
A_L		$L+1$	$1, 2, \cdots, L$	I	$\mathbb{Z}_2$
$D_{\ell+2}$ (ℓ even)		$2\ell+2$	$1, 3, \cdots, 2\ell+1, \ell+1$	I	$\mathbb{Z}_2$
$D_{\ell+2}$ (ℓ odd)		$2\ell+2$	$1, 3, \cdots, 2\ell+1, \ell+1$	II/$A_{2\ell+1}$	$\mathbb{Z}_2$
E_6		12	$1, 4, 5, 7, 8, 11$	I	$\mathbb{Z}_2$
E_7		18	$1, 5, 7, 9, 11, 13, 17$	II/D_{10}	1
E_8		30	$1, 7, 11, 13, 17, 19, 23, 39$	I	1

Figure 1: A-D-E graphs corresponding to the Dynkin diagrams of the classical A-D-E simply laced Lie algebras. The nodes associated with the identity and the fundamental are shown by $*$, $\Box$ respectively. Also shown are the Coxeter numbers g, exponents Exp(G), the Type I or II, the parent graphs $H \neq G$ and the diagram automorphism group Γ. The D_4 graph is exceptional having the automorphism group $\mathbb{S}_3$.

On an infinitely long cylinder, the Hilbert space $\mathcal{H}$ can be expressed as a finite sum of the irreducible representations of the tensor product of the left and the right chiral algebra

$$\mathcal{H} = \bigoplus Z_{j\,\bar{j}} \mathcal{V}_j \otimes \bar{\mathcal{V}}_{\bar{j}} \tag{2.1}$$

where $Z_{j\,\bar{j}}$ are non-negative integer multiplicities. The diagonal subset of the labels $(j, \bar{j})$ of $Z_{j\,\bar{j}}$

$$\mathcal{E} = \{(j, \alpha) | \alpha = 1, \ldots, Z_{j\,j}\} \tag{2.2}$$

is given by the exponents of the Dynkin diagram G of A-D-E type, where α is the multiplicity index.

2.1 Twisted Boundary Conditions on a Torus

In this section, we briefly recall the results on the twisted boundary conditions. Further details are given in [14, 15, 16, 20].

For a WZW model associated with the Dynkin diagram G, the twisted boundary states x on the torus are labeled by the nodes of the Ocneanu graph $x \in \tilde{G}$ [21]. The corresponding twisted operators X_x commute with the Virasoro algebra

$$[L_n, X_x] = [\bar{L}_n, X_x] = 0 \tag{2.3}$$

The Hilbert space with a twisted boundary x inserted is given by

$$\mathcal{H}_x = \bigoplus_{i,\bar{i} \in \mathcal{I}} \tilde{V}_{i,\bar{i}^*;\,x}{}^{1}\, \mathcal{V}_i \otimes \bar{\mathcal{V}}_{\bar{i}} \tag{2.4}$$

where $\tilde{V}_{i,\bar{i}^*;\,x}{}^{1}$ are non-negative integers. The twisted partition functions are labeled by

$$x = (a, b, \kappa) \in (H, \bar{H}, \mathbb{Z}_2) \tag{2.5}$$

where H $(\bar{H})$ is the left (right) chiral copy of the parent graph of G and $\kappa = 1, 2$ labels the $\mathbb{Z}_2$ automorphisms of G (figure 1).

The twisted partition functions in the direct channel are given by the toric matrices $P^{(\kappa)}_{ab}$

$$Z_{(a,b,\kappa)}(q) = \sum_{i,\bar{i}} [P^{(\kappa)}_{ab}]_{i\bar{i}}\; \chi_i(q)\, \chi_{\bar{i}}(\overline{q}), \tag{2.6}$$

$$[P^{(\kappa)}_{ab}]_{i\bar{i}} = \sum_{c \in T_\kappa} n^{(H)c}_{ia}\, n^{(H)\zeta(c)}_{\bar{i}b} \tag{2.7}$$

and $[P^{(\kappa)}_{ab}]_{i\bar{i}} = \tilde{V}_{i,\bar{i}^*;\,x}{}^{1}$. Let us explain the notations in (2.7). $n^{(H)}_i = (n^{(H)b}_{ia})$ are the fused adjacency matrices of the Type I parent graph H of G. The fused adjacency matrices $n_i \equiv n^{(G)}_i$ of a graph G are defined by the $sl(2)$ recurrence relation

$$n_1 = \mathrm{I}, \qquad n_2 = G, \qquad n_{i+1} = n_2\, n_i - n_{i-1} \quad \text{for} \;\; 2 < i < g - 1, \tag{2.8}$$

where $n_2 = G$ is the adjacency matrix of the graph G. T_κ are the following subsets of A_{g-1}

$$T_1 \quad = \quad \begin{cases} \{1,2,\ldots,L\}, & G = A_L \\ \{1,3,5,\ldots,2\ell-1,2\ell\}, & G = D_{2\ell} \\ \{1,2,3,\ldots,4\ell-1\}, & G = D_{2\ell+1} \\ \{1,5,6\}, & G = E_6 \\ \{1,3,5,7,9,10\}, & G = E_7 \\ \{1,7\}, & G = E_8 \end{cases} \tag{2.9}$$

and

$$T_2 \quad = \quad \begin{cases} \{2,4,\ldots,2\ell-2\}, & G = D_{2\ell} \\ T_1, & \text{otherwise} \end{cases} \tag{2.10}$$

The involutive twist ζ is the identity for Type I theories but for Type II theories has the action

$$\zeta = \begin{cases} s \mapsto 4\ell - s, \;\; s = 2,4,\ldots,2\ell-2, & G = D_{2\ell+1} \\ \{1,3,5,7,9,10\} \mapsto \{1,9,5,7,3,10\}, & G = E_7 \end{cases} \tag{2.11}$$

In particular, $Z_{(1,1,1)}$ is the modular invariant partition function with no twist.

2.2 Cylinder Boundary Conditions

The complete set of conformal boundary conditions $|a\rangle$ for $sl(2)$ models on the cylinder are labeled by the nodes of the graph $a \in G$ [22]. They are linear combinations of the Ishibashi states $|j\rangle\rangle_B$ [23]

$$|a\rangle = \sum_{j\in\mathcal{E}} B_a^j \, |j\rangle\rangle_B \,, \qquad B_a^j = \frac{\psi_a^j}{\sqrt{S_1^j}} \tag{2.12}$$

where $S_i^j = \sqrt{\frac{2}{g}} \sin\frac{\pi i j}{g}$ is the modular matrix and the expressions of the coefficients ψ_a^j can be found in the appendix of [22].

The partition functions in the transverse channel are given by

$$A_{b|a}(\tilde q) = \langle b|\tilde q^{\frac{1}{2}(L_0+\bar L_0-\frac{c}{12})}|a\rangle = \sum_{j\in\mathcal{E}} B_a^j (B_b^j)^* \chi_j(\tilde q) \tag{2.13}$$

and in the direct channel

$$A_{b|a}(q) = \sum_{i\in\mathcal{I}} n_{i\,a}{}^b \chi_i(q) \tag{2.14}$$

where $n_{i\,a}{}^b$ are non-negative integers given by (2.8). The direct and the transverse channels are related by the modular transformation S. Comparing (2.14) with (2.13) shows that $n_{i\,a}{}^b$ satisfy the Verlinde-like formula:

$$n_{i\,a}{}^b = \sum_{j\in\mathcal{E}} S_i^j B_a^j (B_b^j)^* = \sum_{j\in\mathcal{E}} \frac{S_i^j}{S_1^j} \psi_a^j (\psi_b^j)^* \tag{2.15}$$

For a cylinder with boundaries a and b and a twisted operator x inserted, the twisted partition function is [15]

$$A_{b|a}^{(x)}(q) = \sum_{i\in\mathcal{I}} (\tilde n_x \cdot n_i)_a{}^b \chi_i(q) \tag{2.16}$$

where $x \in \tilde{G}$ and $\tilde{n}_x$ is the corresponding $|G|\times|G|$ non-negative integer matrix representation. The formulae of $\tilde{n}_x$ are derived in [24] and their explicit expressions are shown in [14].

Note that $A^{(x)}_{b|a}(q)$ can be expressed as a linear combination of the non-twisted partition functions (2.14) since the twisted operator can propagate to the boundary to produce an integrable boundary condition [14].

3 Open Descendants

The complete classification for the open descendants of the modular invariant partition functions for the $sl(2)$ models has been carried out by Pradisi, Sagnotti and Stanev [4, 5]. In this section, we consider the open descendants of twisted boundary conditions.

For an $sl(2)$ model, the crosscap states $|\kappa\rangle$ are labeled by the $\mathbb{Z}_2$ automorphisms of the graph G, with $\kappa = 1, 2$. These $\mathbb{Z}_2$ automorphisms correspond to the simple currents [25] of the Ocneanu algebra $\tilde{G}$. Similar to the boundary states (2.12), the crosscap states are linear combinations of the Ishibashi states $|j\gg_C$

$$|\kappa\rangle = \sum_{j\in\mathcal{E}} \Gamma^{(\kappa)}_j |j\gg_C \tag{3.1}$$

where the crosscap coefficients $\Gamma^{(\kappa)}_j$ will be determined in section 3.3. Note that, unlike the case of the boundary states $|a\rangle$ in which different boundary states can be present simultaneously in a conformal field theory, only one type of crosscap state is allowed [26] due to crosscap constraints.

3.1 Klein bottle

To construct a closed non-oriented CFT on the Klein bottle from the closed oriented theory on the torus, let us start from (2.4). On the Klein bottle, due to the non-orientability, the left chiral fields exchange with the right chiral fields. Essentially all the non-diagonal fields in the Hilbert space (2.4) are projected out, and only the diagonal terms survive. Furthermore, for a CFT on the Klein bottle to be consistent with the non-orientable topology, the corresponding twisted partition function on the torus must be symmetric, i. e. $\tilde{V}_{i,j;x}{}^1 = \tilde{V}_{j,i;x}{}^1$.

On the Klein bottle, the twisted boundary conditions are labeled by $a \in H$, where H is the parent graph of G. It suffices to consider the case of two twisted operators inserted. In the transverse channel, the Klein bottle is viewed as a cylinder with two crosscaps at the ends. The transverse channel twisted Klein bottle partition function with twisted operators a and b inserted is given by

$$K^{(\kappa)}_{(a,b)}(\tilde{q}) = \sum_{j\in\mathcal{E}} \Gamma^{(\kappa)}_j \Gamma^{(\kappa)*}_j \frac{\psi^{(H)j}_a}{\psi^{(H)j}_1} \left(\frac{\psi^{(H)j}_b}{\psi^{(H)j}_1}\right)^* \chi_j(\tilde{q}) \tag{3.2}$$

where $\psi^{(H)j}_a$ are the boundary coefficients of the parent graph H appeared in (2.12). On the other hand, in the direct channel, the twisted partition function is

$$K^{(\kappa)}_{(a,b)}(q) = \sum_{i\in\mathcal{I}} K^{(\kappa)b}_{ia} \chi_i(q) \tag{3.3}$$

where $K_{ia}^{(\kappa)b}$ are the integer Klein bottle coefficients. The two channels are related by the modular transformation S [2], so that $K_{ia}^{(\kappa)b}$ equals

$$K_{ia}^{(\kappa)b} = \sum_{j\in\mathcal{E}} S_i^j \Gamma_j^{(\kappa)} \Gamma_j^{(\kappa)*} \frac{\psi_a^{(H)j}}{\psi_1^{(H)j}} \left(\frac{\psi_b^{(H)j}}{\psi_1^{(H)j}}\right)^* \tag{3.4}$$

The Klein bottle coefficients $K_{ia}^{(\kappa)b}$ satisfy the following property

$$K_{ia}^{(\kappa)b} = K_{ib}^{(\kappa)a} = K_{i\,ab}^{(\kappa)1} = \sum_{c\in H} \hat{N}_{ac}^{(H)b} K_{ic}^{(\kappa)1} \tag{3.5}$$

where $\hat{N}_a^{(H)} = (\hat{N}_{ab}^{(H)c})$ are the graph fusion matrices of the parent graph H given by the Verlinde-like formula

$$\hat{N}_{ab}{}^c = \sum_{j\in\mathcal{E}} \frac{\psi_a^j \psi_b^j (\psi_c^j)^*}{\psi_*^j} \tag{3.6}$$

where $*$ denotes the fundamental node of the graph (figure 1). In general, the graph fusion matrices of a graph G form the graph fusion algebra of G

$$\hat{N}_a \hat{N}_b = \sum_{c\in G} \hat{N}_{ab}{}^c \hat{N}_c \tag{3.7}$$

In particular, for a Type I graph H, its graph fusion matrices are of non-negative integer value.

For simplicity, we consider a single twisted operator inserted on the Klein bottle. The direct channel twisted partition $K_{(a,1)}^{(\kappa)}(q)$ is projected [3] from the torus counterpart $Z_{(a,a,1)}(q)$. The complete non-oriented closed string partition function is half of the sum of the Klein bottle and the torus partition functions in the direct channel

$$Z_{(a,\kappa)}^{\text{non-oriented}} = \frac{1}{2}(Z_{(a,a,1)}^{\text{torus}} + K_{(a,1)}^{(\kappa)}) \tag{3.8}$$

with non-negative integer coefficients. Hence, together with the fact that the Klein bottle coefficients are projected from the torus counterparts, they must satisfy the following constraints

$$|K_{ia}^{(\kappa)1}| \leq [P_{aa}^{(1)}]_{ii} \qquad \text{and} \qquad K_{ia}^{(\kappa)1} \equiv [P_{aa}^{(1)}]_{ii} \bmod 2 \tag{3.9}$$

3.2 Möbius strip

On the Möbius strip, one works with the "tilde" character $\tilde{\chi}(q)$. Recall that the character $\chi(q)$ can be expressed as a polynomial with an overall phase factor

$$\chi_i(q) = q^{h_i - c/24} \sum_k d_i(k) q^k \tag{3.10}$$

[3]The exception is $K_{(2\ell-1,1)}^{(2)} = K_{(2\ell,1)}^{(2)}$ for $D_{2\ell}$ whose torus counterpart is $Z_{(2\ell-1,2\ell,1)}$. We will discuss this from the lattice point of view in section 4.2.1.

The "tilde" character $\tilde{\chi}(q)$ is defined by mapping $q \to e^{i\pi}q$ but with the overall phase factor unchanged

$$\tilde{\chi}_i(q) = q^{h_i - c/24} \sum_k (-1)^k d_i(k) q^k \tag{3.11}$$

In the transverse channel, the Möbius strip is viewed as a cylinder with a crosscap and a boundary at the ends. The transverse channel partition function is given by

$$M_a^{(\kappa)}(\tilde{q}) = \sum_{j \in \mathcal{E}} B_a^j \Gamma_j^{(\kappa)} \tilde{\chi}_j(\tilde{q}) \tag{3.12}$$

whereas the direct channel partition function is

$$M_a^{(\kappa)}(q) = \sum_{i \in \mathcal{I}} M_{ia}^{(\kappa)} \tilde{\chi}_j(q) \tag{3.13}$$

where $M_{ia}^{(\kappa)}$ are integer Möbius strip coefficients. The two channels are transformed by the modular matrix P [2, 12]

$$P = \sqrt{T} S T^2 S \sqrt{T} \tag{3.14}$$

which satisfies $P^2 = C$, where C is the conjugation matrix $C_{ij} = \delta_{j,i^*}$. In (3.14), $\sqrt{T}$ is defined as $(\sqrt{T})_{ij} = e^{i\pi(h_i - c/24)} \delta_{i,j}$.

Then the Möbius strip coefficients equal

$$M_{ia}^{(\kappa)} = \sum_{j \in \mathcal{E}} P_i^j B_a^j \Gamma_j^{(\kappa)} \tag{3.15}$$

The direct channel partition function $M_a^{(\kappa)}(q)$ is projected from the cylinder counterpart $A_{a|a}^{(\kappa)}(q)$, where $\kappa = 1$ is an identity and $\kappa = 2$ denotes the simple current of $\tilde{G}$ which corresponds to the $\mathbb{Z}_2$ symmetry of the graph G. Explicitly,

$$A_{b|a}^{(\kappa)}(q) = \sum_{i \in \mathcal{I}} (\sigma^{\kappa - 1} \cdot n_i)_a{}^b \chi_i(q) \tag{3.16}$$

where σ is the order 2 permutation matrix which encodes the $\mathbb{Z}_2$ automorphism. For convenience, we use the notation $\sigma(a)$ to denote the action of the $\mathbb{Z}_2$ automorphism on the node a of G.

The complete non-oriented open string partition function is half of the sum of the cylinder and the Möbius strip partition functions

$$Z_a^{\text{open}\,(\kappa)} = \frac{1}{2}(A_{a|a}^{(\kappa)} \pm M_a^{(\kappa)}) \tag{3.17}$$

where the sign of $M_a^{(\kappa)}$ cannot be determined from CFT [27]. Thus the Möbius strip coefficients are subject to the following constraints:

$$|M_{ia}^{(\kappa)}| \leq (\sigma^{\kappa-1} \cdot n_i)_a{}^a \qquad \text{and} \qquad M_{ia}^{(\kappa)} \equiv (\sigma^{\kappa-1} \cdot n_i)_a{}^a \bmod 2 \tag{3.18}$$

3.3 Derivation of the crosscap coefficients from Möbius strip

In this section, we follow the method in [9] to compute the crosscap coefficients. The idea of the method is to find solutions to the Möbius strip with vacuum boundary that are consistent with the integrality and positivity conditions of the Klein bottle (3.9) and of the Möbus strip (3.18). Once the Möbius strip coefficients with vacuum boundary are obtained, one can take an inverse of P in (3.15) to get the crosscap coefficients.

In general, for a Möbius strip with boundary state a, we multiply both sides of (3.15) by P to obtain the expressions for $\Gamma_k^{(\kappa)}$

$$\sum_{i\in\mathcal{I}} P_k^i M_{ia}^{(\kappa)} = \sum_{j\in\mathcal{E}} \delta_{k,j} B_a^j \Gamma_j^{(\kappa)} = \sum_{\alpha} B_a^{(k,\alpha)} \Gamma_{(k,\alpha)}^{(\kappa)} \tag{3.19}$$

where α are the multiplicities of the exponents $k \in \mathcal{E}$. Note that for $sl(2)$ WZW models, $\alpha = 1$ except for the case of $k = g/2$ for $D_{2\ell}$ which is of degeneracy 2. Shortly we will see that $\Gamma_k^{(\kappa)} = 0$ for $k \notin \mathcal{E}$.

For a cylinder with vacuum boundary state, the partition function is

$$A_{1|1}^{(\kappa)}(q) = \sum_{i\in\mathcal{I}} (\sigma^{\kappa-1}\cdot n_i)_1{}^1 \chi_i(q) \tag{3.20}$$

The vacuum boundary is non-degenerate so that $(\sigma^{\kappa-1}\cdot n_i)_1{}^1 = 0, 1$. Positivity of the Möbius strip (3.18) requires that the Möbius strip coefficients with vacuum boundary state must satisfy

$$M_{i1}^{(\kappa)} = \eta_i^{(\kappa)} (\sigma^{\kappa-1}\cdot n_i)_1{}^1\,, \qquad \eta_i^{(\kappa)} = \pm 1 \tag{3.21}$$

Thus,

$$\Gamma_k^{(\kappa)} = \frac{1}{B_1^k} \sum_{i\in\mathcal{I}} P_k^i M_{i1}^{(\kappa)} = \frac{\sqrt{S_1^k}}{\psi_1^k} \sum_{i\in\mathcal{I}} \eta_i^{(\kappa)}\, (\sigma^{\kappa-1}\cdot n_i)_1{}^1 P_k^i\,, \qquad k \neq \frac{g}{2} \text{ when } G = D_{2\ell} \tag{3.22}$$

Later, we will show that for $k = \frac{g}{2}$ when $G = D_{2\ell}$

$$\Gamma_{\frac{g}{2}+}^{(\kappa)} = \Gamma_{\frac{g}{2}-}^{(\kappa)} = \frac{1}{2} \frac{\sqrt{S_1^{\frac{g}{2}}}}{\psi_1^{\frac{g}{2}}} \sum_{i\in\mathcal{I}} \eta_i^{(\kappa)}\, (\sigma^{\kappa-1}\cdot n_i)_1{}^1 P_{\frac{g}{2}}^i\,, \qquad G = D_{2\ell} \tag{3.23}$$

In the following section, we will determine the signs of $\eta_i^{(\kappa)}$ up to an overall sign of $\Gamma_k^{(\kappa)}$ by positivity of the Klein bottle coefficients (3.9) and consistency with the solutions of $\Gamma_k^{(\kappa)}$ obtaining from boundary states other than the vacuum.

Before we proceed, we list the properties of P that will be used later on. For $s\ell(2)$ WZW models, P is given by

$$P_i^j = \frac{2}{\sqrt{g}} \sin(\frac{\pi i j}{2g})\, \delta_{(i+j+g) \bmod 2, 0} \tag{3.24}$$

and for g even

$$(-1)^{\frac{g+2}{2}} \sum_{\substack{j=1\\ j=4n+1}}^{g-1} P_k^j = \frac{1}{2}\frac{S^k_{g/2}}{S^k_1}\left\{P_k^1 + (-1)^{\frac{g+2}{2}} P_k^{g-1}\right\} \tag{3.25}$$

$$(-1)^{\frac{g+2}{2}} \sum_{\substack{j=1\\ j=4n+3}}^{g-1} P_k^j = -\frac{1}{2}\frac{S^k_{g/2}}{S^k_1}\left\{P_k^1 - (-1)^{\frac{g+2}{2}} P_k^{g-1}\right\} \tag{3.26}$$

with $S^k_{g/2} = (-1)^{\frac{k-1}{2}}\sqrt{\frac{2}{g}}$ for k odd. Note that both sides of (3.25) and (3.26) are zero for k even.

3.3.1 A_{g-1}

In the A_{g-1} case, there are two simple currents, namely, 1 and σ, where $\sigma = N_{g-1} = (\delta_{i,g-j})$.

$$n_{i1}{}^1 = \delta_{i,1} \qquad \text{and} \qquad (\sigma \cdot n_i)_1{}^1 = n_{i\,\sigma(1)}{}^1 = \delta_{i,g-1} \tag{3.27}$$

so that

$$\Gamma_k^{(1)} = \frac{\eta}{\sqrt{S_1^k}}\, P_k^1 \qquad \text{and} \qquad \Gamma_k^{(2)} = \frac{\eta}{\sqrt{S_1^k}}\, P_k^{g-1} \tag{3.28}$$

where $\eta = \pm 1$ is an overall sign and is undetermined. The non-twisted Klein bottle partition functions are

$$K^{(1)}_{(1,1)}(q) = \sum_{j=1}^{g-1}(-1)^{j-1}\chi_j(q) \qquad \text{and} \qquad K^{(2)}_{(1,1)}(q) = \sum_{j=1}^{g-1}\chi_j(q) \tag{3.29}$$

3.3.2 $D_{2\ell+1}$

The parent graph of $D_{2\ell+1}$ is A_{g-1} with Coxeter number $g = 4\ell$. The $\mathbb{Z}_2$ automorphism of $D_{2\ell+1}$ graph is endowed in the intertwiners $\sigma = n_{g-1}$.

For both $\kappa = 1, 2$ cases,

$$n_{i1}{}^1 = n_{i\,\sigma(1)}{}^1 = \delta_{i,1} + \delta_{i,g-1} \tag{3.30}$$

and

$$\psi_1{}^k = \begin{cases} (-1)^{\frac{k-1}{2}}\sqrt{2}S_1^k & k \neq \frac{g}{2} \\ 0 & k = \frac{g}{2} \end{cases} \tag{3.31}$$

so that

$$\Gamma_k^{(\kappa)} = \frac{(-1)^{\frac{k-1}{2}}}{\sqrt{2S_1^k}}\,\eta(P_k^1 + \tilde{\eta}^{(\kappa)}P_k^{g-1})\,, \qquad k \neq \frac{g}{2} \tag{3.32}$$

and we need to determine the sign of $\tilde{\eta}^{(\kappa)} = \eta^{(\kappa)}_{g-1}\eta^{(\kappa)}_1$. Note that $\Gamma^{(\kappa)}_{\frac{g}{2}}$ is not determined in (3.32) since $\psi_1{}^{\frac{g}{2}} = 0$ and (3.22) does not apply.

Now we repeat the above procedure, not starting from the vacuum boundary state but from the state $\frac{g}{2}^{+}$ (which corresponds to the node $2\ell+1$ in figure 1). The action of σ on the nodes at the fork is $\sigma(\frac{g}{2}^{\pm}) = \frac{g}{2}^{\mp}$, and

$$n_{i\frac{g}{2}^{+}}{}^{\frac{g}{2}^{+}} = \sum_{\substack{j=1 \\ j=4n+1}}^{g-1} \delta_{i,j} \qquad \text{and} \qquad n_{i\frac{g}{2}^{-}}{}^{\frac{g}{2}^{+}} = \sum_{\substack{j=1 \\ j=4n+3}}^{g-1} \delta_{i,j} \tag{3.33}$$

and

$$\psi_{\frac{g}{2}^{+}}{}^{k} = \begin{cases} \frac{(-1)^{\frac{k-1}{2}}}{\sqrt{2}} S_{\frac{g}{2}}{}^{k} = \frac{1}{\sqrt{g}} & k \neq \frac{g}{2} \\ \frac{1}{\sqrt{2}} & k = \frac{g}{2} \end{cases} \tag{3.34}$$

Thus we have another expressions for $\Gamma_k^{(\kappa)}$

$$\Gamma_k^{(\kappa)} = \sqrt{g S_1^k} \sum_{\substack{j=1 \\ j=4n-1+2\kappa}}^{g-1} P_k^j \, \eta_j^{(\kappa)} \tag{3.35}$$

and it follows that $\Gamma_{\frac{g}{2}}^{(\kappa)} = 0$ since $P_{\frac{g}{2}}^{j} = 0$ for j odd.

Equating (3.32) and (3.35) and comparing them with the identities (3.25) and (3.26), we arrive at

$$\begin{aligned} \Gamma_k^{(1)} &= \frac{(-1)^{\frac{k-1}{2}}}{\sqrt{2S_1^k}} \, \eta \, (P_k^1 - P_k^{g-1}) = -\eta \sqrt{g S_1^k} \sum_{\substack{j=1 \\ j=4n+1}}^{g-1} P_k^j & (3.36) \\ &= (-1)^{\frac{(k+2)^2-1}{8}} \, \eta \, \frac{P_k^{\frac{g}{2}-1}}{\sqrt{S_1^k}} & (3.37) \end{aligned}$$

and

$$\begin{aligned} \Gamma_k^{(2)} &= \frac{(-1)^{\frac{k-1}{2}}}{\sqrt{2S_1^k}} \, \eta \, (P_k^1 + P_k^{g-1}) = \eta \sqrt{g S_1^k} \sum_{\substack{j=1 \\ j=4n+3}}^{g-1} P_k^j & (3.38) \\ &= -(-1)^{\frac{(k+2)^2-1}{8}} \, \eta \, \frac{P_k^{\frac{g}{2}+1}}{\sqrt{S_1^k}} & (3.39) \end{aligned}$$

The non-twisted partition functions are

$$K_{(1,1)}^{(1)}(q) = \sum_{\substack{j=1 \\ j \text{ odd}}}^{g-1} \chi_j(q) - \chi_{\frac{g}{2}}(q) \qquad \text{and} \qquad K_{(1,1)}^{(2)}(q) = \sum_{\substack{j=1 \\ j \text{ odd}}}^{g-1} \chi_j(q) + \chi_{\frac{g}{2}}(q) \tag{3.40}$$

3.3.3 $D_{2\ell}$

$D_{2\ell}$ is a Type I graph with Coxeter number $g = 4\ell - 2$. As in the $D_{2\ell+1}$ case, $n_{i1}{}^{1}$ and $n_{i\sigma(1)}{}^{1}$ are given by (3.30). However, unlike the $D_{2\ell+1}$ case, σ is not endowed neither in the

graph fusion matrices of $D_{2\ell}$ nor in the intertwiners with A_{g-1}. In fact, σ is the non-faithful nimrep of the $\mathbb{Z}_2$ simple current of the Ocneanu algebra $\tilde{D}_{2\ell}$.

The coefficients ψ_1^k are given by

$$\psi_1{}^k = \begin{cases} \sqrt{2}S_1^k & k \neq \frac{g}{2} \\ S_1^{\frac{g}{2}} & k = \frac{g}{2} \end{cases} \tag{3.41}$$

so that

$$\Gamma_k^{(\kappa)} = \frac{1}{\sqrt{2S_1^k}}\,\eta(P_k^1 + \tilde{\eta}^{(\kappa)} P_k^{g-1})\,, \qquad k \neq \frac{g^{\pm}}{2} \tag{3.42}$$

$$\Gamma_{\frac{g}{2}^+}^{(\kappa)} + \Gamma_{\frac{g}{2}^-}^{(\kappa)} = \frac{1}{\sqrt{S_1^{\frac{g}{2}}}}\,\eta(P_{\frac{g}{2}}^1 + \tilde{\eta}^{(\kappa)} P_{\frac{g}{2}}^{g-1}) \tag{3.43}$$

To determine the sign of $\tilde{\eta}^{(\kappa)}$ and the crosscap coefficients for $k = \frac{g}{2}^{\pm}$, we repeat the procedure from the boundary state $\frac{g}{2}^+$ as we did in the $D_{2\ell+1}$ case.

$$\psi_{\frac{g}{2}^+}{}^k = \begin{cases} \frac{1}{\sqrt{2}} S_{\frac{g}{2}}{}^k = \frac{(-1)^{\frac{k-1}{2}}}{\sqrt{g}} & k \neq \frac{g}{2} \\ \frac{1}{2}(\, S_{\frac{g}{2}}^{\frac{g}{2}} \pm i\sqrt{(-1)^{\frac{g+2}{4}}}\,) & k = \frac{g}{2}^{\pm} \end{cases} \tag{3.44}$$

and we obtain different expressions for the crosscap coefficients. Comparing with the identities (3.25) and (3.26), we arrive at the results

$$\Gamma_k^{(1)} = \frac{2^{-\frac{1}{2}\delta_{k,g/2}}}{\sqrt{2S_1^k}}\,\eta\,(P_k^1 + P_k^{g-1}) = \eta(-1)^{\frac{k-1}{2}} 2^{-\frac{1}{2}\delta_{k,g/2}} \sqrt{gS_1^k} \sum_{\substack{j=1 \\ j=4n+1}}^{g-1} P_k^j \tag{3.45}$$

and

$$\Gamma_k^{(2)} = \frac{2^{-\frac{1}{2}\delta_{k,g/2}}}{\sqrt{2S_1^k}}\,\eta\,(P_k^1 - P_k^{g-1}) = -\eta(-1)^{\frac{k-1}{2}} 2^{-\frac{1}{2}\delta_{k,g/2}} \sqrt{gS_1^k} \sum_{\substack{j=1 \\ j=4n+3}}^{g-1} P_k^j \tag{3.46}$$

The non-twisted partition functions are

$$K_{(1,1)}^{(1)}(q) = \sum_{\substack{j=1 \\ j \text{ odd} \\ j\neq g/2}}^{g-1} \chi_j(q) + 2\chi_{\frac{g}{2}}(q) = \sum_{\substack{a=1 \\ a \text{ odd} \\ a\neq 2\ell-1}}^{2\ell} \hat{\chi}_a(q) + \hat{\chi}_{2\ell-1}(q) + \hat{\chi}_{2\ell}(q) \tag{3.47}$$

$$K_{(1,1)}^{(2)}(q) = \sum_{\substack{j=1 \\ j \text{ odd} \\ j\neq g/2}}^{g-1} \chi_j(q) = \sum_{\substack{a=1 \\ a \text{ odd} \\ a\neq 2\ell-1}}^{2\ell} \hat{\chi}_a(q) + \hat{\chi}_{2\ell-1}(q) - \hat{\chi}_{2\ell}(q) \tag{3.48}$$

where $\hat{\chi}_a(q)$ are the extended characters defined as

$$\hat{\chi}_a(q) = \sum_{i\in A_{g-1}} n_{i1}{}^a \chi_i(q) \tag{3.49}$$

Note that the $\kappa = 2$ case is not discussed in [5] since the Klein bottle coefficients do not satisfy the fusion rules of the extended chiral algebra, namely,

$$\hat{K}_a \hat{K}_b \hat{K}_c > 0 \qquad \text{if} \quad \hat{N}_{ab}{}^c \neq 0 \tag{3.50}$$

where $\hat{K}_a$ are the Klein bottle coefficients in terms of the extended characters. The debate as to whether the constraint (3.50) is necessary for the Klein bottle coefficients is not settled. Refer to [3] for more detail.

Finally, we remark that the D_4 model possesses $\mathbb{S}_3$ symmetry which generates 6 simple currents. However, it turns out that they do not satisfy integrality except for the cases we consider here.

3.3.4 E_6

E_6 has two simple currents, the identity and $\sigma = \hat{N}_5$.

$$n_{i1}{}^1 = \delta_{i,1} + \delta_{i,7} \qquad \text{and} \qquad n_{i\,\sigma(1)}{}^1 = n_{i\,5}{}^1 = \delta_{i,5} + \delta_{i,11} \tag{3.51}$$

By integrality,

$$\Gamma_k^{(1)} = \frac{\sqrt{S_1^k}}{\psi_1^k}\, \eta\, (P_k^1 - P_k^7) \tag{3.52}$$

and

$$\Gamma_k^{(2)} = \frac{\sqrt{S_1^k}}{\psi_1^k}\, \eta\, (P_k^5 + P_k^{11}) \tag{3.53}$$

The non-twisted Klein bottle partition functions are

$$K^{(1)}_{(1,1,1)}(q) = \chi_1 - \chi_4 + \chi_5 + \chi_7 - \chi_8 + \chi_{11} = \hat{\chi}_1 + \hat{\chi}_5 - \hat{\chi}_6 \tag{3.54}$$

and

$$K^{(2)}_{(1,1,1)}(q) = \chi_1 + \chi_4 + \chi_5 + \chi_7 + \chi_8 + \chi_{11} = \hat{\chi}_1 + \hat{\chi}_5 + \hat{\chi}_6 \tag{3.55}$$

3.3.5 E_7

E_7 is a Type II graph whose parent graph is D_{10}.

$$n_{i1}{}^1 = \delta_{i,1} + \delta_{i,9} + \delta_{i,17} \tag{3.56}$$

By integrality,

$$\Gamma_k^{(1)} = \frac{\sqrt{S_1^k}}{\psi_1^k}\, \eta\, (P_k^1 + P_k^9 + P_k^{17}) \tag{3.57}$$

and the non-twisted Klein bottle partition function is

$$K^{(1)}_{(1,1,1)}(q) = (\chi_1 + \chi_{17}) + (\chi_5 + \chi_{13}) + (\chi_7 + \chi_{11}) + \chi_9 = \hat{\chi}_1 + \hat{\chi}_5 + \hat{\chi}_7 + \hat{\chi}_9 \tag{3.58}$$

where $\hat{\chi}$ in (3.58) are the extended characters of the parent graph D_{10}.

3.3.6 E_8

$$n_{i1}{}^{1} = \delta_{i,1} + \delta_{i,11} + \delta_{i,19} + \delta_{i,29} \tag{3.59}$$

By integrality,

$$\Gamma_k^{(1)} = \frac{\sqrt{S_1^k}}{\psi_1^k}\,\eta\,(P_k^1 - P_k^{11} - P_k^{19} - P_k^{29}) \tag{3.60}$$

and the non-twisted Klein bottle partition function is

$$K_{(1,1,1)}^{(1)}(q) = (\chi_1 + \chi_{11} + \chi_{19} + \chi_{29}) + (\chi_7 + \chi_{13} + \chi_{17} + \chi_{23}) = \hat{\chi}_1 + \hat{\chi}_7 \tag{3.61}$$

3.4 Unitary minimal models

One can generalize the method in the previous section to minimal models. In particular, we are interested in the unitary cases whose realizations in statistical mechanics are known. These will be discussed in section 4.

The unitary minimal models are labeled by a pair of graphs (A_{g-2}, G). The central charge is $c = 1 - \frac{6}{g(g-1)}$ and the conformal weights are given by the Kac formula subject to the Kac symmetry

$$h_{r,s} \equiv h_{g-1-r,g-s} = \frac{[gr-(g-1)s]^2 - 1}{4g(g-1)} \tag{3.62}$$

For later convenience, we define the set of Kac indices $\mathcal{I}$ to be

$$\mathcal{I} = \begin{cases} \{(r,s) \mid r \text{ odd}\ ,\ 1 \le s \le g-1\}\,, & g \text{ even} \\ \{(r,s) \mid 1 \le r \le g-2,\ s \text{ odd}\ \}\,, & g \text{ odd} \end{cases} \tag{3.63}$$

and the set of exponents $\mathcal{E}$

$$\mathcal{E} = \begin{cases} \{(r,a) \mid r \text{ odd}\ , a \in \mathrm{Exp}(G)\}\,, & g \text{ even} \\ \mathcal{I}\,, & g \text{ odd} \end{cases} \tag{3.64}$$

Note that g is always even for D and E cases.

• **Twisted boundary conditions on the torus**

On the torus, the twisted partition functions are labeled by the tensor product Ocneanu graph $A_{g-2} \otimes \tilde{G}$

$$(r,x) = (r,a,b,\kappa) \in (A_{g-2}, H, \bar{H}, \mathbb{Z}_2) \tag{3.65}$$

and they are given by

$$Z_{(r,a,b,\kappa)}(q) = \sum_{(r',s),(r'',\bar{s})} N_{r\,r'}^{(A_{g-2})r''}\,[P_{ab}^{(\kappa)}]_{s\bar{s}}\,\chi_{(r',s)}(q)\,\chi_{(r'',\bar{s})}(\overline{q}), \tag{3.66}$$

• **Cylinder boundary conditions**

The conformal boundary conditions for the unitary minimal models are labeled by the nodes of tensor product graph $(r,a) \in A_{g-1} \otimes G$ with the symmetry

$$(r,a) \sim (g-1-r, \gamma(a))\,, \qquad \text{where } \gamma = \begin{cases} \sigma_G & \text{for } G = A, D_{2\ell+1}, E_6 \\ 1 & \text{otherwise} \end{cases} \tag{3.67}$$

The boundary states are expressed as linear sums of the Ishibashi states

$$|(r,a)\!>= \sum_{(r',s')\in\mathcal{E}} \frac{\Psi_{(r,a)}^{\ (r',s')}}{\sqrt{S_{(r,s)}^{(1,1)}}}\,|r',s'\gg \tag{3.68}$$

where

$$\Psi_{(r,a)}^{\ (r',s')} = \begin{cases} \sqrt{2}S_r^{(g-1)r'}\,\psi_a^{\ s'} & r,r' \text{ odd},\ s'\in \mathrm{Exp}(G) \text{ for } g \text{ even} \\ S_{(r,s)}^{(r',s')} & (r,s),(r',s')\in\mathcal{I} \text{ for } g \text{ odd} \end{cases} \tag{3.69}$$

and $S_{(r,s)}^{(r',s')} = (-1)^{(r+s)(r'+s')}\sqrt{2}S_r^{(g-1)r'}\,S_s^{(g)s'}$ is the modular matrix, where $S^{(g-1)}$ and $S^{(g)}$ are the modular matrices of $SU(2)_k$ at levels $g-3$ and $g-2$ respectively.

The cylinder partition function with boundaries (r_1,a) and (r_2,b) is

$$\begin{aligned} A_{(r_1,a)|(r_2,b)}(q) &= \sum_{(r,s)\in\mathcal{I}} n_{rs;(r_1,a)}^{\quad\ (r_2,b)}\,\chi_{(r,s)}(q) && (3.70)\\ &= \sum_{r,s} N_{r\,r_1}^{(A_{g-2})r_2}\,n_{s\,a}^{\ \ b}\,\chi_{(r,s)}(q) && (3.71) \end{aligned}$$

in which the non-negative coefficients are given by

$$\begin{aligned} n_{rs;(r_1,a)}^{\quad\ (r_2,b)} &= \sum_{(r',s')\in\mathcal{E}} S_{(r,s)}^{(r',s')}\,\frac{\Psi_{(r,a)}^{\ (r',s')}}{\sqrt{S_{(1,1)}^{(r',s')}}}\,\frac{\Psi_{(r,b)}^{\ (r',s')}}{\sqrt{S_{(1,1)}^{(r',s')}}} && (3.72)\\ &= N_{r\,r_1}^{(A_{g-2})r_2}\,n_{s\,a}^{\ \ b} + N_{g-1-r\ \ r_1}^{(A_{g-2})\qquad r_2}\,n_{g-s\ \ a}^{\quad\ \ b} && (3.73) \end{aligned}$$

where $N_r^{(A_{g-2})}$ and n_s are the fused adjacency matrices of A_{g-2} and G respectively.

• Klein bottle

Under the symmetry (3.67), on the tensor product graph $A_{g-2}\otimes G$, there are two independent $\mathbb{Z}_2$ automorphisms, namely, the identity and $\sigma=(1,\sigma_G)$. These $\mathbb{Z}_2$ automorphisms corresponds to the simple currents of the tensor Ocneanu algebra $A_{g-2}\otimes\tilde{G}$ subject to quantum symmetries.

The Klein bottle partition functions are labeled by $(r,a,b)\in(A_{g-2},H,H)$

$$K_{(r,a,b)}^{(\kappa)}(q) = \sum_{(r'',s'')\in\mathcal{I}} K_{(r'',s'');(r,a,b)}^{(\kappa)}\,\chi_{(r'',s'')}(q) \tag{3.74}$$

where $K_{(r'',s'');(r,a,b)}^{(\kappa)}$ are integer coefficients

$$K_{(r'',s'');(r,a,b)}^{(\kappa)} = \sum_{(r',s')\in\mathcal{E}} S_{(r'',s'')}^{(r',s')}\,\Gamma_{(r,s)}^{(\kappa)}\Gamma_{(r,s)}^{(\kappa)*}\,\frac{S_r^{(g-1)r'}}{S_1^{(g-1)r'}}\,\frac{\psi_a^{(H)s'}}{\psi_1^{(H)s'}}\left(\frac{\psi_b^{(H)s'}}{\psi_1^{(H)s'}}\right)^* \tag{3.75}$$

• Möbius strip

For unitary minimal models with Coxeter number g, the $|\mathcal{I}| \times |\mathcal{I}|$ modular matrix P is the sum of the direct products of the modular matrices $P^{(g-1)}$ and $P^{(g)}$ of $SU(2)_{g-3}$ and $SU(2)_{g-2}$

$$P_{(r_1,s_1)}^{(r_2,s_2)} = \sigma_{r1,s1}\,(\,\sigma_{r2,s2}\,P_{r_1}^{(g-1)r_2}P_{s_1}^{(g)s_2} \;+\; \sigma_{g-1-r2,g-s2}\,P_{r_1}^{(g-1)g-1-r_2}P_{s_1}^{(g)g-s_2}\,) \tag{3.76}$$

where

$$\sigma_{r,s} = \begin{cases} -1\,, & (r-s)\bmod 4 \equiv 2 \\ 1\,, & \text{otherwise} \end{cases} \tag{3.77}$$

Note that for the choice of $\mathcal{I}$ as defined in (3.63), the first term of (3.76) vanishes.

The Möbius strip partition functions are labeled by $(r,a) \in (A_{g-2}, G)$

$$M_{(r,a)}^{(\kappa)}(q) = \sum_{i\in\mathcal{I}} M_{(r'',s'')}^{(\kappa)}\tilde{\chi}_{(r,s)}(q) \tag{3.78}$$

where $M_{(r'',s'')}^{(\kappa)}$ are the integer coefficients

$$M_{(r'',s'')}^{(\kappa)} = \sum_{(r',s')\in\mathcal{E}} P_{(r'',s'')}^{(r',s')}\,\frac{\Psi_{(r,a)}^{\;(r',s')}}{\sqrt{S_{(1,1)}^{(r',s')}}}\,\Gamma_{(r',s')}^{(\kappa)} \tag{3.79}$$

In addition to the symmetry (3.67), $(r,a) \sim (r,\gamma(a))$ since ${\psi_a}^j$ equals ${\psi_{\gamma(a)}}^j$ up to a sign [22].

In the following we list the crosscap coefficients of the unitary A-D-E models and their non-twisted partition functions on the Klein bottle.

3.4.1 (A_{g-2}, A)

$$\Gamma_{(r,s)}^{(1)} = \frac{\eta}{\sqrt{S_{(1,1)}^{(r,s)}}}\,P_{(r,s)}^{(1,1)}\,, \qquad \Gamma_{(r,s)}^{(2)} = \frac{\eta}{\sqrt{S_{(1,1)}^{(r,s)}}}\,P_{(r,s)}^{(1,g-1)} \tag{3.80}$$

and

$$K_{(1,1,1)}^{(1)}(q) = \sum_{(r,s)\in\mathcal{I}} \chi_{(r,s)}(q)\,, \qquad K_{(1,1,1)}^{(2)}(q) = \sum_{(r,s)\in\mathcal{I}} (-1)^{r+s}\chi_{(r,s)}(q) \tag{3.81}$$

3.4.2 $(A_{g-2}, D_{2\ell+1})$

$$\Gamma_{(r,s)}^{(\kappa)} = \frac{(-1)^{\frac{s-1}{2}}}{\sqrt{2S_{(1,1)}^{(r,s)}}}\,\eta\,(P_{(r,s)}^{(1,1)} - (-1)^{\kappa}\,P_{(r,s)}^{(1,g-1)}) \tag{3.82}$$

and

$$K_{(1,1,1)}^{(\kappa)}(q) = \sum_{\substack{r=1\\ r\text{ odd}}}^{g-1}\;\sum_{\substack{s=1\\ s\text{ odd}}}^{g-1} \chi_{(r,s)}(q) - (-1)^{\kappa}\,\chi_{r,\frac{g}{2}}(q) \tag{3.83}$$

3.4.3 $(A_{g-2}, D_{2\ell})$

$$\Gamma^{(\kappa)}_{(r,s)} = \frac{2^{-\frac{1}{2}\delta_{s,g/2}}}{\sqrt{2S^{(r,s)}_{(1,1)}}}\,\eta\,(P^{(1,1)}_{(r,s)} - (-1)^{\kappa}\,P^{(1,g-1)}_{(r,s)}) \tag{3.84}$$

and

$$K^{(1)}_{(1,1,1)}(q) = \sum_{\substack{r=1\\ r \text{ odd}}}^{g-1}\ \sum_{\substack{s=1\\ s \text{ odd}\\ s\neq g/2}}^{g-1} \chi_{(r,s)}(q) + 2\chi_{r,\frac{g}{2}}(q) \tag{3.85}$$

$$K^{(2)}_{(1,1,1)}(q) = \sum_{\substack{r=1\\ r \text{ odd}}}^{g-1}\ \sum_{\substack{s=1\\ s \text{ odd}\\ s\neq g/2}}^{g-1} \chi_{(r,s)}(q) \tag{3.86}$$

3.4.4 (A_{10}, E_6)

$$\Gamma^{(\kappa)}_{(r,s)} = \frac{\sqrt{S^{(r,s)}_{(1,1)}}}{\Psi^{(r,s)}_{(1,1)}}\,\eta\,(P^{(1,1)}_{(r,s)} - (-1)^{\kappa}\,P^{(1,7)}_{(r,s)}) \tag{3.87}$$

and

$$\begin{aligned} K^{(\kappa)}_{(1,1,1)}(q) &= \sum_{\substack{r=1\\ r \text{ odd}}}^{g-1} (\chi_{(r,1)} + \chi_{(r,7)}) + (\chi_{(r,5)} + \chi_{(r,11)}) - (-1)^{\kappa}(\chi_{(r,4)} + \chi_{(r,8)}) & (3.88)\\ &= \sum_{\substack{r=1\\ r \text{ odd}}}^{g-1} \hat{\chi}_{(r,1)}(q) + \hat{\chi}_{(r,5)}(q) - (-1)^{\kappa}\hat{\chi}_{(r,6)}(q) & (3.89)\end{aligned}$$

3.4.5 (A_{16}, E_7)

$$\Gamma^{(1)}_{(r,s)} = \frac{\sqrt{S^{(r,s)}_{(1,1)}}}{\Psi^{(r,s)}_{(1,1)}}\,\eta\,(P^{(1,1)}_{(r,s)} + P^{(1,9)}_{(r,s)} + P^{(1,17)}_{(r,s)}) \tag{3.90}$$

and

$$\begin{aligned} K^{(\kappa)}_{(1,1,1)}(q) &= \sum_{\substack{r=1\\ r \text{ odd}}}^{g-1} (\chi_{(r,1)} + \chi_{(r,17)}) + (\chi_{(r,5)} + \chi_{(r,13)}) + (\chi_{(r,7)} + \chi_{(r,11)}) + \chi_{(r,9)} & (3.91)\\ &= \sum_{\substack{r=1\\ r \text{ odd}}}^{g-1} \hat{\chi}_{(r,1)}(q) + \hat{\chi}_{(r,5)}(q) + \hat{\chi}_{(r,7)}(q) + \hat{\chi}_{(r,9)}(q) & (3.92)\end{aligned}$$

where $\hat{\chi}_{(r,a)}(q)$ are the extended characters of the parent graph (A_{16}, D_{10}).

3.4.6 (A_{28}, E_8)

$$\Gamma^{(1)}_{(r,s)} = \frac{\sqrt{S^{(r,s)}_{(1,1)}}}{\Psi^{(r,s)}_{(1,1)}}\,\eta\,(P^{(1,1)}_{(r,s)} + P^{(1,11)}_{(r,s)} + P^{(1,19)}_{(r,s)} - P^{(1,29)}_{(r,s)}) \tag{3.93}$$

and

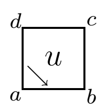
$$K^{(\kappa)}_{(1,1,1)}(q) = \sum_{\substack{r=1 \\ r \text{ odd}}}^{g-1} (\chi_{(r,1)}+\chi_{(r,11)}+\chi_{(r,19)}+\chi_{(r,29)}) + (\chi_{(r,7)}+\chi_{(r,13)}+\chi_{(r,17)}+\chi_{(r,23)}) \tag{3.94}$$

$$= \sum_{\substack{r=1 \\ r \text{ odd}}}^{g-1} \hat{\chi}_{(r,1)}(q) + \hat{\chi}_{(r,7)}(q) \tag{3.95}$$

4 Lattice Realizations of the Klein Bottle and Möbius Strip Amplitudes

It is well known that the $s\ell(2)$ unitary minimal theories are realized as the continuum scaling limit of critical A-D-E lattice models [28]. Theses solvable lattice models are associated with the Dynkin diagrams G of a simply laced Lie algebra of A, D, or E type. The spin states a, b, c, d are nodes of the graph G and the spins of adjacent sites on the lattice must be adjacent nodes on the graph. The (unfused) Boltzmann face weight is given by

$$W^{11}\left(\begin{array}{cc|c} d & c & \\ a & b & u \end{array}\right) =$$



$$= s(\lambda - u)\,\delta_{ac} + s(u)\sqrt{\frac{\psi_a \psi_c}{\psi_b \psi_d}}\,\delta_{bd} \tag{4.1}$$

and zero otherwise. Here, u is the spectral parameter with $0 < u < \lambda$, $\lambda = \frac{\pi}{g}$, $s(u) = \frac{\sin(u)}{\sin(\lambda)}$ and ψ_a are the entries of the Perron-Frobenius eigenvector of the adjacency matrix G.

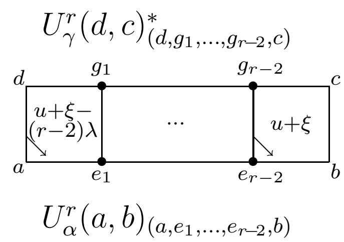
The CFT on a torus with defect lines is realized as an insertion of seams on the lattice. There are three kinds of seams, namely, r-type, a-type and automorphism seam. Generally, the r-type and a-type seams are simply fused Boltzmann weights. The product of the seams are identically labelled by $(r, x) = (r, a, b, \kappa) \in (A_{g-2}, H, \bar{H}, \mathbb{Z}_2)$ as in (3.65), where r denotes the r-type seam, and a and b denote the respective left-chiral and right-chiral a-type seams, and κ the automorphism seam. This composite seam realizes the corresponding twisted boundary conditions.

4.1 Seam

Generally, the r-type and a-type seams are simply fused Boltzmann weights. In this section, we quickly recall the definition of seams. For the details on the construction of seams, we refer to [14, 29].

The r-type seam $W_{(r,1)}\left(\begin{smallmatrix} d & \gamma & c \\ a & \alpha & b \end{smallmatrix}\,\middle|\, u,\xi\right)$ is obtained by fusing $r-1$ faces

$$W_{(r,1)}\left(\begin{matrix} d & \gamma & c \\ a & \alpha & b \end{matrix}\,\middle|\, u,\xi\right) = \frac{1}{s_{-1}^{r-2}(u+\xi)}$$

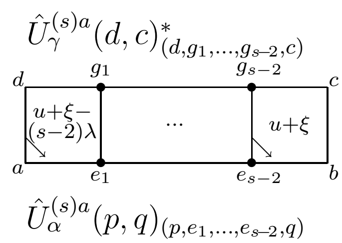


(4.2)

where $s_i^p(u) = \prod_{j=0}^{p-1} s(u+(i-j)\lambda)$, and $\boldsymbol{U}^r(a,b)$ are the fusion vectors which are the normalized eigenvectors of the idempotent fusion projector $P^r(a,b)$ with non-zero eigenvalues [29], and $\alpha = 1,2,\ldots,{n_{ra}}^b$ and $\gamma = 1,2,\ldots,{n_{rc}}^d$ are the bond variables labeling these eigenvectors. The solid dots indicate that the spins are summed out.

The left-chiral a-type seam is the braid limit of the fused face weight

$$W_{(1,a)}\left(\begin{matrix} d & \gamma & c \\ p & \alpha & q \end{matrix}\right) = \lim_{\xi\to i\infty} \frac{\exp(-i\frac{(g+1)(s-1)}{2}\lambda)}{s(u+\xi)\, s_{-1}^{r-2}(u+\xi)}$$



(4.3)

where $\hat{U}^{(s)a}(d,c)$ are the fusion vectors of the $\hat{N}$ fusion projector $\hat{P}^{(s)a}(d,c)$ which are obtained from decomposition of $P^s(d,c)$ [14]

$$P^s(d,c) = \sum_{a\in G} {n_{s1}}^a \hat{P}^{(s)a}(d,c) \tag{4.4}$$

The right-chiral a-type seam is given by the braid limit $\xi \to -i\infty$. Since the automorphism seam does not play a role on non-orientable topologies, we do not discuss it here.

4.2 Torus and Klein bottle transfer matrices

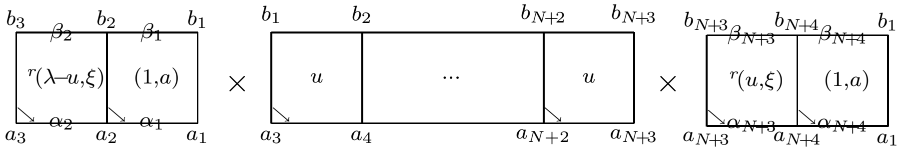
In this section, we construct transfer matrices on the Klein bottle. We start by constructing a transfer matrix on a torus. A torus is obtained by joining the ends of a cylinder. The boundaries are periodic. On the other hand, the Klein bottle is achieved by twisting one end of the cylinder before gluing them together. Correspondingly, for the partition function on the Klein bottle, a flip operator is inserted before taking the trace of the transfer matrix. This transfer matrix must be consistent with the flip operator as we shall see below.

The entries of the single-row torus transfer matrix $\boldsymbol{T}_{(r,a)}(u,\xi)$ with a pair of (r,a) seams

inserted are defined as

$$\langle \boldsymbol{a},\boldsymbol{\alpha}|\ \boldsymbol{T}_{(r,a)}(u,\xi)\ |\boldsymbol{b},\boldsymbol{\beta}\rangle =$$

$$\sqrt{\frac{\psi_{a_1}\psi_{b_3}}{\psi_{a_3}\psi_{b_1}}}$$



$$= \sqrt{\frac{\psi_{a_1}\psi_{b_3}}{\psi_{a_3}\psi_{b_1}}}\ W_{(r,1)}\left(\begin{matrix} b_3 & \beta_2 & b_2 \\ & & \\ a_3 & \alpha_2 & a_2 \end{matrix}\ \middle|\ \lambda-u,\xi\right) W_{(1,a)}\left(\begin{matrix} b_2 & \beta_1 & b_1 \\ & & \\ a_2 & \alpha_1 & a_1 \end{matrix}\right)\times$$
$$\prod_{i=3}^{N+2} W\left(\begin{matrix} b_i & & b_{i+1} \\ & & \\ a_i & & a_{i+1} \end{matrix}\ \middle|\ u\right)\times W_{(r,1)}\left(\begin{matrix} b_{N+3} & \beta_{N+3} & b_{N+4} \\ & & \\ a_{N+3} & \alpha_{N+3} & a_{N+4} \end{matrix}\ \middle|\ u,\xi\right) W_{(1,a)}\left(\begin{matrix} b_{N+4} & \beta_{N+4} & b_1 \\ & & \\ a_{N+4} & \alpha_{N+4} & a_1 \end{matrix}\right) \tag{4.5}$$

Note that the path of the left-hand seams is reversed. It will be clear shortly that such construction is necessary in order to make the transfer matrix consistent with the Klein bottle geometry. One can show by crossing symmetry that the flipped a-type seam on the left hand side is equivalent to the complex conjugate of a non-flipped a-type seam. This is the reason the corresponding partition function is ambichiral. The transfer matrix forms a commuting family thanks to the generalized Yang-Baxter equation and thus the lattice model is integrable.

The flip operator $\boldsymbol{F}$ which interchanges spins on the left and on the right is given by

$$\langle \boldsymbol{a},\boldsymbol{\alpha}|\ \boldsymbol{F}\ |\boldsymbol{b},\boldsymbol{\beta}\rangle = \delta_{\alpha_1,\beta_{N+4}}\,\delta_{\alpha_2,\beta_{N+3}}\,\delta_{\alpha_{N+3},\beta_2}\,\delta_{\alpha_{N+4},\beta_1}\prod_{i=1}^{N+4}\delta_{a_i,b_{N+6-i}} \tag{4.6}$$

and the $\mathbb{Z}_2$ symmetry operator $\boldsymbol{R}$

$$\langle \boldsymbol{a},\boldsymbol{\alpha}|\ \boldsymbol{R}\ |\boldsymbol{b},\boldsymbol{\beta}\rangle = \delta_{\alpha_1,\beta_1}\,\delta_{\alpha_2,\beta_2}\,\delta_{\alpha_{N+3},\beta_{N+3}}\,\delta_{\alpha_{N+4},\beta_{N+4}}\prod_{i=1}^{N+4}\delta_{a_i,\sigma(b_i)} \tag{4.7}$$

i. e. the $\mathbb{Z}_2$ automorphism acts on spins at each site locally. Clearly $\boldsymbol{F}$ and $\boldsymbol{R}$ commute and $\boldsymbol{F}^2 = \boldsymbol{R}^2 = 1$.

One can show that

$$\boldsymbol{F}\,\boldsymbol{T}_{(r,a)}(u,\xi) = \boldsymbol{T}_{(r,a)}(\lambda-u,\xi)\,\boldsymbol{F} \tag{4.8}$$

Note that in order for (4.8) to hold, the flip of the left-hand seam and the existence of the gauge factor in (4.5) are necessary. Moreover, $[\boldsymbol{R},\boldsymbol{T}_{(r,a)}]=0$ with an appropriate normalization of the fusion vectors

$$U^{(s)a}_{\alpha}(d,c)_{(d,g_1,\ldots,g_{s-2},c)} \ = \ \hat{U}^{(s)a}_{\alpha}(\sigma(d),\sigma(c))_{(\sigma(d),\sigma(g_1),\ldots,\sigma(g_{s-2}),\sigma(c))} \tag{4.9}$$

Such normalization is always possible since the adjacency condition $G_{a,b} = G_{\sigma(a),\sigma(b)}$ and the Boltzmann weight is invariant under the action of the $\mathbb{Z}_2$ automorphism on the spins

$$W^{11}\left(\begin{array}{cc|c} d & c & \\ a & b & u \end{array}\right) = W^{11}\left(\begin{array}{cc|c} \sigma(d) & \sigma(c) & \\ \sigma(a) & \sigma(b) & u \end{array}\right) \tag{4.10}$$

The double-row transfer matrix is defined as

$$\boldsymbol{D}_{(r,a)}(u,\xi) = \boldsymbol{T}_{(r,a)}(u,\xi)\,\boldsymbol{T}_{(r,a)}(\lambda-u,\xi) \tag{4.11}$$

so that $\boldsymbol{F}$ and $\boldsymbol{D}_{(r,a)}(u,\xi)$ commute for any value of u and ξ.

We claim that the partition functions on the Klein bottle are given by [4]

$$K^{(\kappa)}_{(r,a)\,MN}(u,\xi) = \mathrm{Tr}\,\boldsymbol{F}^{(\kappa)}\,(\boldsymbol{D}_{(r,a)}(u,\xi))^M\,, \qquad a \neq 2\ell-1, 2\ell \text{ for } D_{2\ell} \text{ when } \kappa=2 \tag{4.12}$$

where $\boldsymbol{F}^{(\kappa)} = \boldsymbol{F}\boldsymbol{R}^{\kappa-1}$ and M is odd. For $\kappa=1$, $\boldsymbol{F}^{(1)}$ is simply a geometric flip in the usual sense. On the other hand, for $\kappa=2$, it involves a flip in the spin configuration space. This flip corresponds to the $\mathbb{Z}_2$ symmetry of the Dynkin diagrams G. Such spin-state flip for the Ising model on non-orientable surfaces was known and its interpretation was given in [12]. The correctness of (4.12) is verified numerically in section 5.

4.2.1 $D_{2\ell}$ case

The continuum scaling limit of the transfer matrix $\boldsymbol{T}_{(r,a)}$, with a pair of $(1,a)$ seams inserted as in (4.5), gives the conformal partition function $Z_{(1,a,a,1)}(q)$ on a torus. For the $D_{2\ell}$ cases, whose Ocneanu algebras are non-commutative, the quantum symmetries of $D_{2\ell}$ for the nodes at the fork of the graph G give

$$Z_{(r,2\ell-1,\kappa)} = \begin{cases} Z_{(r,1,2\ell-1,\kappa)} & \ell \text{ odd} \\ Z_{(r,1,2\ell,\kappa)} & \ell \text{ even} \end{cases} \tag{4.13}$$

and similarly for the node 2ℓ. This commutation relation is reflected on the graph fusion matrices

$$\begin{array}{ll} \hat{N}_{2\ell} \text{ and } \hat{N}_{2\ell-1} \text{ symmetric} & \ell \text{ odd} \\ \hat{N}_{2\ell} = \hat{N}^T_{2\ell-1} & \ell \text{ even} \end{array} \tag{4.14}$$

and $\sigma\,\hat{N}_{2\ell-1} = \hat{N}_{2\ell-1}\,\sigma$.

On the Klein bottle, the partition function (4.12) is not defined for $D_{2\ell}$ when ℓ is even, since the spin path is not admissible under the action of $\boldsymbol{F}^{(2)}$.

To realize the partition function for $K^{(2)}_{(r,2\ell)}(q) = K^{(2)}_{(r,2\ell-1)}(q)$, the partition function for the lattice model is given by

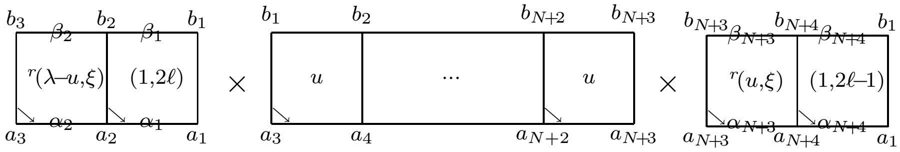
$$K^{(2)}_{(r,2\ell)\,MN}(u,\xi) = \mathrm{Tr}\,(\boldsymbol{F}^{(2)}\,(\boldsymbol{T}'_{r,2\ell}(u,\xi)\,\boldsymbol{T}'_{r,2\ell}(\lambda-u,\xi))^M) \tag{4.15}$$

[4] (4.12) is not valid for A_{even}. See (5.5) in section 5.

where $\boldsymbol{T}'_{r,2\ell}(u,\xi)$ is defined as in (4.5) but with $(1,2\ell)$ and $(1,2\ell-1)$ seams at the left and right ends instead

$$\langle \boldsymbol{a},\boldsymbol{\alpha}|\; \boldsymbol{T}'_{(r,2\ell)}(u,\xi)\;|\boldsymbol{b},\boldsymbol{\beta}\rangle =$$

$$\sqrt{\frac{\psi_{a_1}\psi_{b_3}}{\psi_{a_3}\psi_{b_1}}}$$



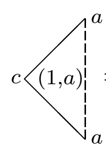
$$= \sqrt{\frac{\psi_{a_1}\psi_{b_3}}{\psi_{a_3}\psi_{b_1}}}\; W_{(r,1)}\begin{pmatrix} b_3 & \beta_2 & b_2 \\ & & \\ a_3 & \alpha_2 & a_2 \end{pmatrix}\!\!\left|\; \lambda-u,\xi \right) W_{(1,2\ell)}\begin{pmatrix} b_2 & \beta_1 & b_1 \\ & & \\ a_2 & \alpha_1 & a_1 \end{pmatrix} \times$$
$$\prod_{i=3}^{N+2} W\left(\begin{matrix} b_i & b_{i+1} \\ & \\ a_i & a_{i+1}\end{matrix}\;\middle|\; u\right) \times W_{(r,1)}\left(\begin{matrix} b_{N+3} & \beta_{N+3} & b_{N+4} \\ & & \\ a_{N+3} & \alpha_{N+3} & a_{N+4}\end{matrix}\;\middle|\; u,\xi\right) W_{(1,2\ell-1)}\begin{pmatrix} b_{N+4} & \beta_{N+4} & b_1 \\ & & \\ a_{N+4} & \alpha_{N+4} & a_1 \end{pmatrix} \tag{4.16}$$

so that it is consistent with the flip $F^{(2)}$.

4.3 Cylinder and Möbius strip transfer matrices

The vacuum boundary condition corresponds to $(r,a)=(1,1)$. The $(1,a)$ boundary weights, for two adjacent nodes of G, c and a (i. e. $G_{ac}\neq 0$) are given by

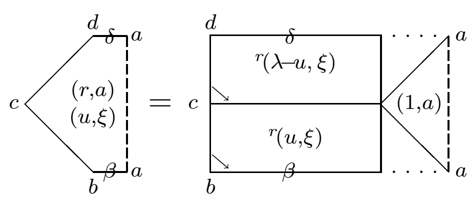
$$B_{(1,a)}\begin{pmatrix} & a \\ c & \\ & a \end{pmatrix} =$$



$$= \frac{\psi_c^{1/2}}{\psi_a^{1/2}} \tag{4.17}$$

An (r,a) boundary weight is given by the action of r-type seams on the $(1,a)$-boundary weight

$$B_{(r,a)}\left(\begin{matrix} & d & \delta \\ c & & \\ & b & \beta \end{matrix}\;\middle|\; u,\xi\right) =$$



(4.18)

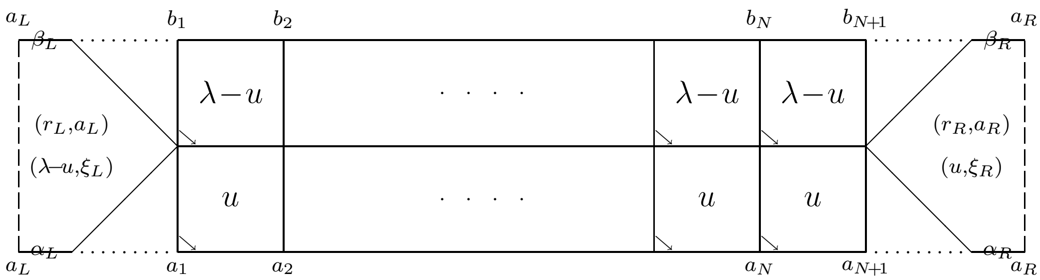
and the left boundary weights are equal to the right boundary up to a gauge factor

$$B_{(r,a)}\left(\begin{matrix} \delta\, d & \\ & c \\ \beta\, b & \end{matrix}\;\middle|\; u,\xi\right) = \sqrt{\frac{\psi_d}{\psi_b}}\; B_{(r,a)}\left(\begin{matrix} & d & \delta \\ c & & \\ & b & \beta \end{matrix}\;\middle|\; u,\xi\right) \tag{4.19}$$

These boundary weights satisfy the boundary Yang-Baxter equation.

The double row transfer matrix is given by

$$\langle \boldsymbol{a}, \boldsymbol{\alpha}|\ \boldsymbol{T}_{(r_L,a_L)|(r_R,a_R)}(u, \xi_L, \xi_R)\ |\boldsymbol{b}, \boldsymbol{\beta}\rangle =$$



(4.20)

One can show that $\boldsymbol{T}$ is symmetric.

Similarly, the flip operator $\boldsymbol{F}$ on the cylinder is given by

$$\langle \boldsymbol{a}, \boldsymbol{\alpha}|\ \boldsymbol{F}\ |\boldsymbol{b}, \boldsymbol{\beta}\rangle = \delta_{\alpha_L,\beta_R}\,\delta_{\alpha_R,\beta_L}\prod_{i=1}^{N+1}\delta_{a_i,b_{N+2-i}} \tag{4.21}$$

and the $\mathbb{Z}_2$ symmetry operator $\boldsymbol{R}$

$$\langle \boldsymbol{a}, \boldsymbol{\alpha}|\ \boldsymbol{R}\ |\boldsymbol{b}, \boldsymbol{\beta}\rangle = \delta_{\alpha_L,\beta_L}\,\delta_{\alpha_R,\beta_R}\prod_{i=1}^{N+1}\delta_{a_i,\sigma(b_i)} \tag{4.22}$$

and $\boldsymbol{F}^{(\kappa)} = \boldsymbol{F}\boldsymbol{R}^{\kappa-1}$.

The Möbius strip partition functions are

$$M^{(\kappa)}_{(r,a)\,MN} = \mathrm{Tr}\,\boldsymbol{F}^{(\kappa)}(\boldsymbol{T}_{(r,\sigma^{\kappa-1}(a))|(r,a)}(u,\xi,\xi))^M \tag{4.23}$$

where M is odd.

5 Numerical Results

The numerics described in this section were carried out using *Mathematica* [30].

The conformal points are $(u,\xi) = (\frac{\lambda}{2}, \xi_c)$, where $\xi_c = \frac{\lambda}{2}(r-2+kg)$ for k odd. At these points, $\boldsymbol{F}^{(\kappa)}$ and $\boldsymbol{T}_{(r,a)}$ commute. Instead of diagonalizing the double-row transfer matrix, we diagonalize $\boldsymbol{F}^{(\kappa)}\boldsymbol{T}_{(r,a)}(\frac{\lambda}{2},\xi_c)$. The reason for doing this is that taking a product of large matrices in *Mathematica* consumes substantial memory. For the same reason, we do not multiply the matrices $\boldsymbol{F}^{(\kappa)}$ and $\boldsymbol{T}_{(r,a)}$ directly. Notice that the action of $\boldsymbol{F}^{(\kappa)}$ on $\boldsymbol{T}_{(r,a)}$ is simply a permutation of the path basis of the transfer matrix. More precisely,

$$\langle \boldsymbol{a}, \boldsymbol{\alpha}|\ \boldsymbol{F}^{(\kappa)}\boldsymbol{T}_{(r,a)}\ |\boldsymbol{b}, \boldsymbol{\beta}\rangle = \langle F^{(\kappa)}(\boldsymbol{a}, \boldsymbol{\alpha})|\ \boldsymbol{T}_{(r,a)}\ |\boldsymbol{b}, \boldsymbol{\beta}\rangle \tag{5.1}$$

where $F^{(\kappa)}(\boldsymbol{a}, \boldsymbol{\alpha})$ denotes the permutation on the path

$$\begin{aligned}(\boldsymbol{a}, \boldsymbol{\alpha}) &= (a_1, a_2, a_3, \cdots, a_{N+3}, a_{N+4}, a_1;\ \alpha_1, \alpha_2, \alpha_3, \alpha_4)\\ F^{(\kappa)}(\boldsymbol{a}, \boldsymbol{\alpha}) &= (\sigma^{\kappa-1}(a_1), \sigma^{\kappa-1}(a_{N+4}), \sigma^{\kappa-1}(a_{N+3}), \cdots,\\ &\qquad \sigma^{\kappa-1}(a_3), \sigma^{\kappa-1}(a_2), \sigma^{\kappa-1}(a_1);\ \alpha_4, \alpha_3, \alpha_2, \alpha_1)\end{aligned} \tag{5.2}$$

We project $\boldsymbol{T}_{(r,a)}(\frac{\lambda}{2},\xi_c)$ onto the eigenbasis of $\boldsymbol{F}^{(\kappa)}$:

$$F^{(\kappa)} = \pm 1 \ : \qquad\qquad \frac{1}{2}(\boldsymbol{F}^{(\kappa)} \pm 1)\, \boldsymbol{T}_{(r,a)}(\frac{\lambda}{2},\xi_c) \tag{5.3}$$

and diagonalize them separately. Thus, the partition function becomes

$$K^{(\kappa)}_{(r,a)\,MN} = \left(\sum_i |T_i^+|^{2M}\right) - \left(\sum_j |T_j^-|^{2M}\right) \tag{5.4}$$

where $T_i^\pm$ are the eigenvalues of (5.3). Since only the diagonal terms of the partition function survive on the Klein bottle, all the non-diagonal terms cancel in (5.4). As we showed in [17], the non-diagonal terms have the characteristic that the zeros of the corresponding eigenvalue functions are not symmetric on the upper and lower half complex plane. Note that for the A_{even} models, the trace of $\boldsymbol{F}^{(2)}\boldsymbol{D}_{(r,a)}$ is zero, since the action of $\mathbb{Z}_2$ automorphism σ interchanges the sublattices of even and odd spins. In these cases, the partition functions are defined as

$$K^{(2)}_{(r,a)\,MN} = \sum_i (T_i^+)^M + \sum_j (T_j^-)^M\,, \qquad M \text{ odd} \tag{5.5}$$

For $0 < u < \lambda$, a critical lattice model gives rise to a conformal field theory in the continuum scaling limit, namely, an $sl(2)$ unitary minimal model. This is manifest in the finite-size corrections to the eigenvalues of the transfer matrices.

For the periodic row transfer matrix $\boldsymbol{T}_{(r,a)}(u,\xi)$ with a pair of seam (r,a) and N faces excluding the seams as defined in (4.5), the eigenvalues are

$$T_n(u) = \exp(-E_n(u)), \quad n = 0,1,2,\dots \tag{5.6}$$

The finite-size corrections to the energies E_n take the form

$$\begin{aligned} E_n(u) &= N f(u) + f_r(u,\xi) \\ &+ \frac{2\pi}{N}\Big(\big(-\frac{c}{12} + h_n + \bar{h}_n + k_n + \bar{k}_n\big)\sin\vartheta + i(h_n - \bar{h}_n + k_n - \bar{k}_n)\cos\vartheta\Big) + o\left(\frac{1}{N}\right) \end{aligned} \tag{5.7}$$

where $f(u)$ is the bulk free energy, $f_r(u,\xi)$ is the boundary free energy due to the r-type seam, c is the central charge, h_n and $\bar{h}_n$ are conformal weights, $k_n, \bar{k}_n \in \mathbb{N}$ label descendent levels and the anisotropy angle ϑ is given by

$$\vartheta = g u \tag{5.8}$$

where g is the Coxeter number.

In the continuum scaling limit $M \gg N \gg 1$, the conformal partition $K^{(\kappa)}_{(r,a)}(q)$ is given by

$$K^{(\kappa)}_{(r,a)}(q) \simeq \exp(N f(u) + f_r(u,\xi_c))\, K^{(\kappa)}_{(r,a)\,MN}(u,\xi_c) \tag{5.9}$$

where $q = \exp(4\pi i\tau) \equiv q^2_{\text{torus}}$ is the modular parameter for the Klein bottle with $\tau = \frac{M}{N}\exp[i(\pi-\vartheta)]$. The free energies were computed by solving a functional equation of the transfer matrices [31] and they were shown in [14].

We carried out the numerics for A_3, A_4, A_5, D_4 and D_5 with seams inserted on the Klein bottle and on the Möbius strip. We fit the sets of data $\{\log T_i \mid N = 2, 4, 6, \ldots\}$ onto a polynomial of the form $a_1 N + a_0 + \frac{a_{-1}}{N} + \cdots \frac{a_{-k}}{N^k}$ and extrapolate it to $1/N \to 0$. By comparing with the expression (5.7), the eigenvalue spectra agree well with the result obtained from the theory of open descendants in section 3. The Möbius strip and Klein bottle partition functions of the D_4 and A_5 models are listed in tables 4–7. In figure 2 and in table 1 we illustrate some typical numerical results for the A_5 case with a $(1,2)$ seam inserted. In figure 3 and in table 2 we show the D_4 case with seam $(1,2)$; in figure 4 and in table 3 we illustrate the D_4 case with seam $(1,3)$ for $\kappa = 2$.

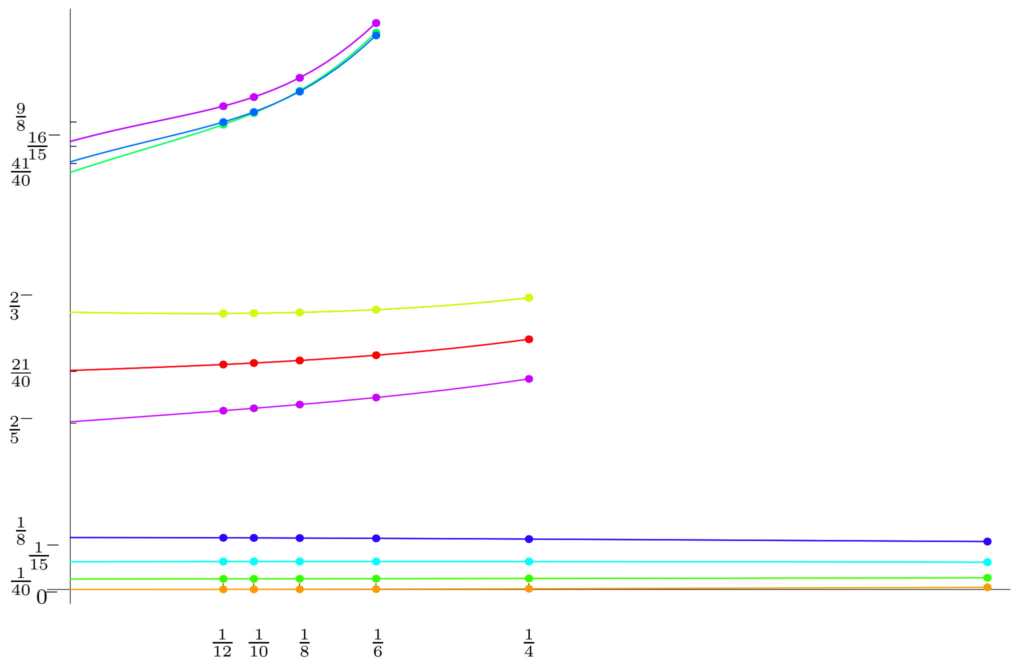


Figure 2: The extrapolated sequences of the first ten energy levels of the minimal (A_4, A_5) model with $(1,2)$ seam on the Klein bottle for $\kappa = 1, 2$. The energy levels corresponding to the terms of negative coefficients in $K^{(2)}$ are marked with a superscript $^-$. The horizontal axis is $1/N$.

A_5 : $(1,2)$										
n	0	1	2	3	4	5	6	7	8	9
Degeneracy	1	2	2	2	1	2	2	2	2	2
Exact	0^-	$\frac{1}{40}$	$\frac{1}{15}^-$	$\frac{1}{8}$	$\frac{2}{5}^-$	$\frac{21}{40}$	$\frac{2}{3}^-$	$\frac{41}{40}$	$\frac{16}{15}^-$	$\frac{9}{8}$
Numerical	10^{-6}	0.0250	0.0667	0.1250	0.4033	0.5260	0.6649	0.9779	0.9988	1.0442
$\lvert$diff.$\rvert$	10^{-6}	10^{-5}	10^{-5}	10^{-5}	0.0033	0.0001	0.0018	0.0471	0.0679	0.0809

Table 1: Numerical values of the exponents of the Klein bottle partition functions $K^{(\kappa)}_{(1,2,1)}$ for (A_4, A_5) minimal model with a $(1,2)$ seam.

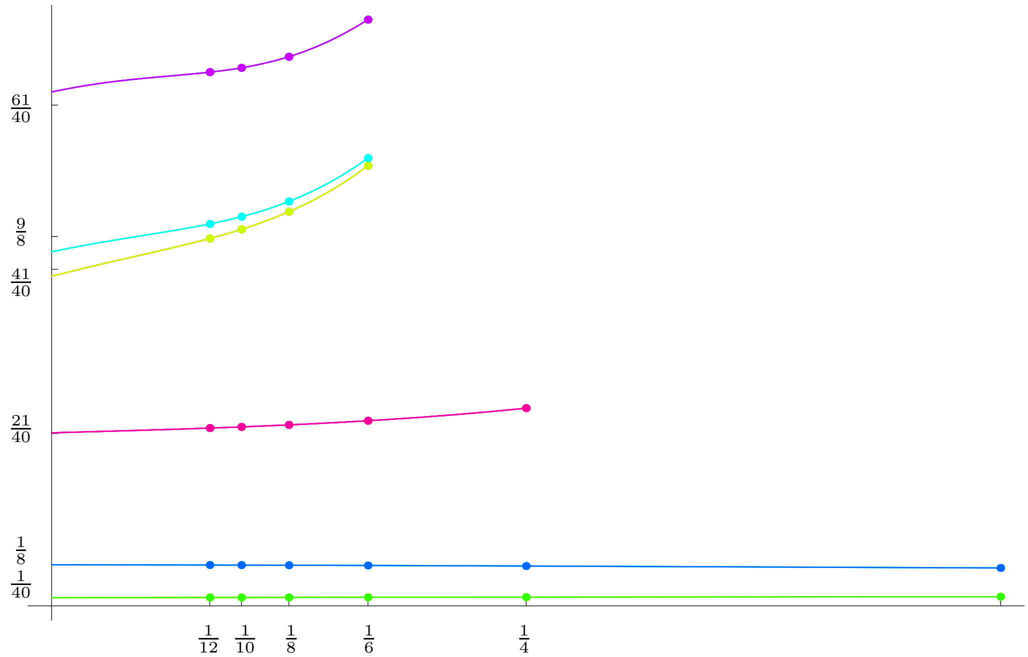


Figure 3: The extrapolated sequences of the first six energy levels of the minimal (A_4, D_4) model with $(1, 2)$ seam on the Klein bottle for $\kappa = 1, 2$. The only difference between the $\kappa = 1, 2$ cases are the degeneracies of the energy levels.

6 Discussion

In this paper we studied A-D-E lattice models on the Klein bottle and on the Möbius strip with defect lines. The non-orientable geometries were achieved by inserting a flip operator before taking the trace of the transfer matrices. The structure of the transfer matrices is not arbitrary. They must be consistent with the non-orientable topologies, namely, the transfer matrices have to be left-right symmetric upon the action of the flip operator.

Furthermore, we showed that in order to realize all the open descendants of the unitary minimal models, not only the flip on the topologies of the lattice but also a flip in the spin configuration space must be applied. This flip in the spin configuration space is the $\mathbb{Z}_2$

$D_4: \quad (1,2)$						
n	0	1	2	3	4	5
Degeneracy (κ=1 / 2)	3/1	3/1	3/1	3/1	3/1	3/1
Exact	$\frac{1}{40}$	$\frac{1}{8}$	$\frac{21}{40}$	$\frac{41}{40}$	$\frac{9}{8}$	$\frac{61}{40}$
Numerical	0.0250	0.1250	0.5260	0.9779	1.0442	1.5102
|diff.|	4.5×10^{-5}	3.3×10^{-5}	9.7×10^{-4}	0.0471	0.0809	0.0148

Table 2: Numerical values of the exponents of the Klein bottle partition functions $K^{(\kappa)}_{(1,2,1)}$ for (A_4, D_4) minimal model with a $(1, 2)$ seam. The degeneracies of the first six energy levels for $\kappa = 1$ and 2 are three and one respectively.

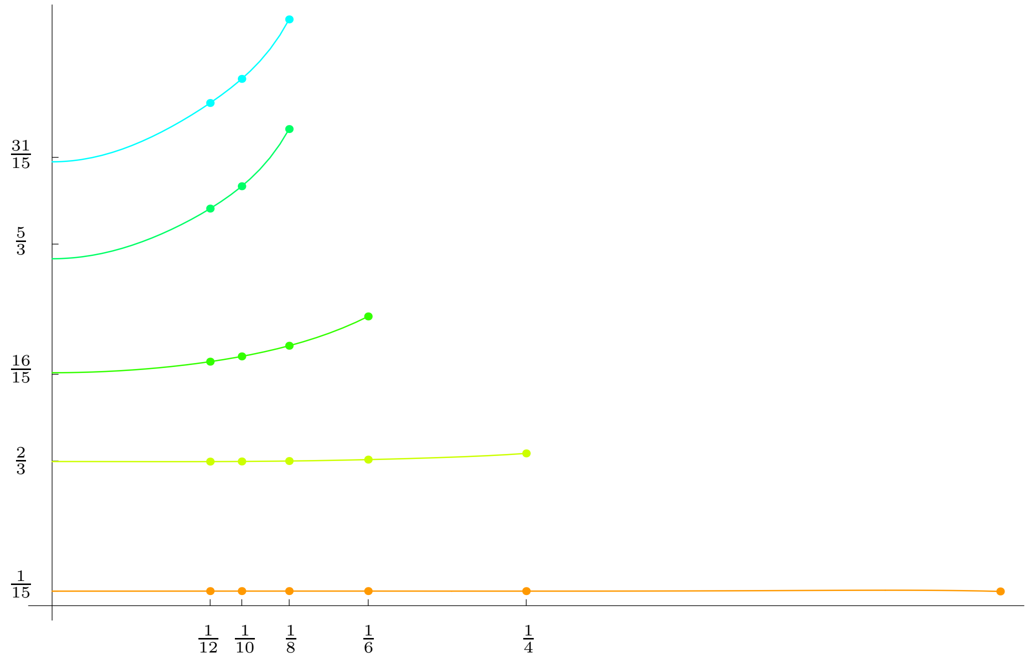


Figure 4: The extrapolated sequences of the first five energy levels of the minimal (A_4, D_4) model with $(1,3)$ seam on the Klein bottle for $\kappa = 2$.

D_4 : $\kappa = 2$. $(1,3)$					
n	0	1	2	3	4
Exact	$\frac{1}{15}$	$\frac{2}{3}$	$\frac{16}{15}$	$\frac{5}{3}$	$\frac{31}{15}$
Numerical	0.0671	0.6646	1.0739	1.5992	2.0640
|diff.|	4.5×10^{-4}	0.0021	0.0072	0.0675	0.0207

Table 3: Numerical values of the exponents of the Klein bottle partition functions $K^{(2)}_{(1,3,1)}$ for (A_4, D_4) minimal model with a $(1,3)$ seam.

automorphism of the graph G, and essentially it corresponds to the simple current of the Ocneanu fusion algebra. We diagonalized the finite-size lattice transfer matrices numerically and verified that the extrapolated partition functions agree well with the prediction of the theory of open descendants.

Finally, we remark that the methods of this paper can be applied to $\mathbb{Z}_k$ parafermionic models and other rational conformal field theories whose lattice realizations are known.

Acknowledgements

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$$
\begin{aligned}
M^{(1)}_{(1,1)} &= \chi_{1,1}+\chi_{1,5} \\
M^{(1)}_{(1,2)} &= \chi_{1,1}+2\chi_{1,3}-\chi_{1,5} \\
M^{(1)}_{(1,3)} &= \chi_{1,1}+\chi_{1,5} \quad = M^{(1)}_{(1,4)} \\
M^{(1)}_{(3,1)} &= \chi_{1,1}+\chi_{1,5}+\chi_{3,1}+\chi_{3,5} \\
M^{(1)}_{(3,2)} &= \chi_{1,1}+2\chi_{1,3}-\chi_{1,5}+\chi_{3,1}+2\chi_{3,3}-\chi_{3,5} \\
M^{(1)}_{(3,3)} &= \chi_{1,1}+\chi_{1,5}+\chi_{3,1}+\chi_{3,5} \quad = M^{(1)}_{(3,4)} \\
M^{(2)}_{(1,1)} &= \chi_{1,1}-\chi_{1,5} \\
M^{(2)}_{(1,2)} &= \chi_{1,1}+\chi_{1,5} \\
M^{(2)}_{(1,3)} &= \chi_{1,3} \quad = M^{(2)}_{(1,4)} \\
M^{(2)}_{(3,1)} &= \chi_{1,1}-\chi_{1,5}+\chi_{3,1}-\chi_{3,5} \\
M^{(2)}_{(3,2)} &= \chi_{1,1}+\chi_{1,5}+\chi_{3,1}+\chi_{3,5} \\
M^{(2)}_{(3,3)} &= \chi_{1,3}+\chi_{3,3} \quad = M^{(2)}_{(3,4)}
\end{aligned}
$$

Table 4: The (A_4, D_4) Möbius strip partition functions of the 3-state Potts model.

$$
\begin{aligned}
K^{(1)}_{(1,1)} &= \chi_{1,1}+2\chi_{1,3}+\chi_{1,5}+\chi_{3,1}+2\chi_{3,3}+\chi_{3,5} \\
K^{(1)}_{(1,2)} &= 3\chi_{1,2}+3\chi_{1,4}+3\chi_{3,2}+3\chi_{3,4} \\
K^{(1)}_{(1,3)} &= \chi_{1,1}+2\chi_{1,3}+\chi_{1,5}+\chi_{3,1}+2\chi_{3,3}+\chi_{3,5} \quad = K^{(1)}_{(1,4)} \\
K^{(1)}_{(3,1)} &= \chi_{1,1}+2\chi_{1,3}+\chi_{1,5}+2\chi_{3,1}+4\chi_{3,3}+2\chi_{3,5} \\
K^{(1)}_{(3,2)} &= 3\chi_{1,2}+3\chi_{1,4}+6\chi_{3,2}+6\chi_{3,4} \\
K^{(1)}_{(3,3)} &= \chi_{1,1}+2\chi_{1,3}+\chi_{1,5}+2\chi_{3,1}+4\chi_{3,3}+2\chi_{3,5} \quad = K^{(1)}_{(3,4)} \\
K^{(2)}_{(1,1)} &= \chi_{1,1}+\chi_{1,5}+\chi_{3,1}+\chi_{3,5} \\
K^{(2)}_{(1,2)} &= \chi_{1,2}+\chi_{1,4}+\chi_{3,2}+\chi_{3,4} \\
K^{(2)}_{(1,3)} &= \chi_{1,3}+\chi_{3,3} \quad = K^{(2)}_{(1,4)} \\
K^{(2)}_{(3,1)} &= \chi_{1,1}+\chi_{1,5}+2\chi_{3,1}+2\chi_{3,5} \\
K^{(2)}_{(3,2)} &= \chi_{1,2}+\chi_{1,4}+2\chi_{3,2}+2\chi_{3,4} \\
K^{(2)}_{(3,3)} &= \chi_{1,3}+2\chi_{3,3} \quad = K^{(2)}_{(3,4)}
\end{aligned}
$$

Table 5: The (A_4, D_4) Klein bottle partition functions of the 3-state Potts model.

$$
\begin{aligned}
M^{(1)}_{(1,1)} &= \chi_{1,1} \\
M^{(1)}_{(1,2)} &= \chi_{1,1}+\chi_{1,3} \\
M^{(1)}_{(1,3)} &= \chi_{1,1}+\chi_{1,3}+\chi_{1,5} \\
M^{(1)}_{(3,1)} &= \chi_{1,1}+\chi_{3,1} \\
M^{(1)}_{(3,2)} &= \chi_{1,1}+\chi_{1,3}+\chi_{3,1}+\chi_{3,3} \\
M^{(1)}_{(3,3)} &= \chi_{1,1}+\chi_{1,3}+\chi_{1,5}+\chi_{3,1}+\chi_{3,3}+\chi_{3,5} \\
M^{(2)}_{(1,1)} &= \chi_{1,5} \\
M^{(2)}_{(1,2)} &= \chi_{1,3}-\chi_{1,5} \\
M^{(2)}_{(1,3)} &= \chi_{1,1}-\chi_{1,3}+\chi_{1,5} \\
M^{(2)}_{(3,1)} &= \chi_{1,5}+\chi_{3,5} \\
M^{(2)}_{(3,2)} &= \chi_{1,3}-\chi_{1,5}+\chi_{3,3}-\chi_{3,5} \\
M^{(2)}_{(3,3)} &= \chi_{1,1}-\chi_{1,3}+\chi_{1,5}+\chi_{3,1}-\chi_{3,3}+\chi_{3,5}
\end{aligned}
$$

Table 6: The (A_4, A_5) Möbius strip partition functions. They satisfy (3.67) and $M^{(\kappa)}_{(r,s)} = M^{(\kappa)}_{(r,6-s)}$.

$$
\begin{aligned}
K^{(1)}_{(1,1)} &= \chi_{1,1}+\chi_{1,2}+\chi_{1,3}+\chi_{1,4}+\chi_{1,5}+\chi_{3,1}+\chi_{3,2}+\chi_{3,3}+\chi_{3,4}+\chi_{3,5} \\
K^{(1)}_{(1,2)} &= \chi_{1,1}+2\chi_{1,2}+2\chi_{1,3}+2\chi_{1,4}+\chi_{1,5}+\chi_{3,1}+2\chi_{3,2}+2\chi_{3,3}+2\chi_{3,4}+\chi_{3,5} \\
K^{(1)}_{(1,3)} &= \chi_{1,1}+2\chi_{1,2}+3\chi_{1,3}+2\chi_{1,4}+\chi_{1,5}+\chi_{3,1}+2\chi_{3,2}+3\chi_{3,3}+2\chi_{3,4}+\chi_{3,5} \\
K^{(1)}_{(3,1)} &= \chi_{1,1}+\chi_{1,2}+\chi_{1,3}+\chi_{1,4}+\chi_{1,5}+2\chi_{3,1}+2\chi_{3,2}+2\chi_{3,3}+2\chi_{3,4}+2\chi_{3,5} \\
K^{(1)}_{(3,2)} &= \chi_{1,1}+2\chi_{1,2}+2\chi_{1,3}+2\chi_{1,4}+\chi_{1,5}+2\chi_{3,1}+4\chi_{3,2}+4\chi_{3,3}+4\chi_{3,4}+2\chi_{3,5} \\
K^{(1)}_{(3,3)} &= \chi_{1,1}+2\chi_{1,2}+3\chi_{1,3}+2\chi_{1,4}+\chi_{1,5}+2\chi_{3,1}+4\chi_{3,2}+6\chi_{3,3}+4\chi_{3,4}+2\chi_{3,5} \\
K^{(2)}_{(1,1)} &= \chi_{1,1}-\chi_{1,2}+\chi_{1,3}-\chi_{1,4}+\chi_{1,5}+\chi_{3,1}-\chi_{3,2}+\chi_{3,3}-\chi_{3,4}+\chi_{3,5} \\
K^{(2)}_{(1,2)} &= -\chi_{1,1}+2\chi_{1,2}-2\chi_{1,3}+2\chi_{1,4}-\chi_{1,5}-\chi_{3,1}+2\chi_{3,2}-2\chi_{3,3}+2\chi_{3,4}-\chi_{3,5} \\
K^{(2)}_{(1,3)} &= \chi_{1,1}-2\chi_{1,2}+3\chi_{1,3}-2\chi_{1,4}+\chi_{1,5}+\chi_{3,1}-2\chi_{3,2}+3\chi_{3,3}-2\chi_{3,4}+\chi_{3,5} \\
K^{(2)}_{(3,1)} &= \chi_{1,1}-\chi_{1,2}+\chi_{1,3}-\chi_{1,4}+\chi_{1,5}+2\chi_{3,1}-2\chi_{3,2}+2\chi_{3,3}-2\chi_{3,4}+2\chi_{3,5} \\
K^{(2)}_{(3,2)} &= -\chi_{1,1}+2\chi_{1,2}-2\chi_{1,3}+2\chi_{1,4}-\chi_{1,5}-2\chi_{3,1}+4\chi_{3,2}-4\chi_{3,3}+4\chi_{3,4}-2\chi_{3,5} \\
K^{(2)}_{(3,3)} &= \chi_{1,1}-2\chi_{1,2}+3\chi_{1,3}-2\chi_{1,4}+\chi_{1,5}+2\chi_{3,1}-4\chi_{3,2}+6\chi_{3,3}-4\chi_{3,4}+2\chi_{3,5}
\end{aligned}
$$

Table 7: The (A_4, A_5) Klein bottle partition functions. The twisted boundary conditions satisfy (3.67) and $K^{(\kappa)}_{(r,s)} = K^{(\kappa)}_{(r,6-s)}$.

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